On the stochastic SIS model with jumps perturbation

Adil Lahrouz*, Adel Settati, Hamza El Mahjour, Akhrif Abdelhadi

*Corresponding author :lahrouzadil@gmail.com

May 7th 2015

Introduction : SIS epidemic model

Infectious diseases are those caused by pathogens (such as viruses, bacteria, epiphytes) or parasites (such as protozoans, worms), and which can spread in the population. The impact of infectious diseases on human and animal is enormous, both in terms of physical and mental sufferings, and social and economic consequences.

Different mathematical models exist to study the progress of an epidemic in a population:
SIR, SIRS, SIS etc.

- S $\rightarrow$ Susceptible
- I $\rightarrow$ Infected

Example : SIS models infections such us common flu

Classic SIS model

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt, \end{cases} \tag{1}$$

where :

- $\mu \rightarrow$ birth rate (coinciding with death rate)
- $\beta \rightarrow$ infection coefficient
- $\gamma \rightarrow$ recovery rate

Constant population $\Rightarrow$ S + I = 1

Stochastic SIS model

- Model proposed by Gray et al. (2011)
  The perturbation : $\beta \rightsquigarrow \beta + \sigma dB(t)$

$$SDE \rightarrow \begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt - \sigma SI dB, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt + \sigma SI dB, \end{cases} \tag{2}$$

- Model proposed in this presentation
  Stochastic perturbation+jumps:

$$\beta \rightsquigarrow \beta + \sigma dB(t) + \int_{\mathbb{Y}} H(z)\widetilde{N}(dt, dz)$$

Where :

- $\widetilde{N}$: represents a compensated Poisson measure
- H : affects of random jumps on β
- $\mathbb{Y}$: measurable subset of $[0, \infty)$

So, equation 2 becomes :

$$\begin{cases} dS(t) = (\mu - \mu S(t) - \beta S(t)I(t) + \gamma I(t))\, dt - \sigma S(t)I(t)dB(t) - \int_{\mathbb{Y}} H(z)S(t)I(t)\widetilde{N}(dt, dz), \\ dI(t) = (-(\mu + \gamma)I(t) + \beta S(t)I(t))\, dt + \sigma S(t)I(t)dB(t) + \int_{\mathbb{Y}} H(z)S(t)I(t)\widetilde{N}(dt, dz), \end{cases} \tag{3}$$

What are the advantages of using jumps ?
1- In reality, the infection coefficient β can change dramatically its value (for example in a population with common flu a surprising change of weather affects quickly the value the spread of infection).So, the model with jumps is more suitable. 2- Bao et al. (2011) proved that convergence rate using jumps has order $1/p$ for any $p \geq 2$ instead of $1/2$ for any $p \geq 2$ without jumps.

Well posedness

Let define the two following sets :

$$\mathbb{R}^2_+ = \{(x_1, x_2) | x_i > 0,\ i = 1, 2\} \qquad \Delta = \left\{ x \in \mathbb{R}^2_+;\ x_1 + x_2 = 1 \right\},$$

The following theorem ensures that our model is well posed.

Theorem

Define $C(z, x) = (1 - H(z)x)\,(1 + H(z)(1 - x))$, *for* $(z, x) \in \mathbb{Y} \times (0, 1)$. *If*

$$\sup_{0<x<1} \int_{\mathbb{Y}} \left(\log \frac{1}{C(z, x)} \right) \nu(dz) < \infty. \tag{4}$$

Then, the set Δ *is almost surely positively invariant by the system* (3), *that is, if* $(S(0), I(0)) \in \Delta$, *then* $\mathbb{P}\left((S(t), I(t)) \in \Delta\right) = 1$ *for all* $t \geq 0$.

Extinction of disease

Since we have $S(t) + I(t) = 1$ we will focus on $I(t)$ and study the following SDE :

$$\begin{aligned} dI(t) &= I(t)\left(\beta(1 - I(t)) - (\mu + \gamma)\right) dt + \sigma I(t)(1 - I(t))dB(t) \\ &\quad + \int_{\mathbb{Y}} H(z)I(t)(1 - I(t))\widetilde{N}(dt, dz). \qquad (5) \\ &\triangleq f_1(I(t))dt + f_2(I(t))dB(t) + \int_{\mathbb{Y}} f_3(I(t), z)\widetilde{N}(dt, dz). \end{aligned}$$

Theorem

Let assumption 4 hold and assume that

$$\sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2 \, \nu(dz) < \infty. \tag{6}$$

and

$$\textit{If } B \leq 0, \quad \textit{then} \quad \limsup_{t\to\infty} \frac{1}{t} \log(I(t)) \leq C \quad a.s.$$

$$\textit{If } B > 0, \quad \textit{then} \quad \limsup_{t\to\infty} \frac{1}{t} \log(I(t)) \leq \frac{B^2}{4A} + C \quad a.s.,$$

where
$A = \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\right)$,
$B = -\beta + \sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)$,
$C = \beta - \left(\mu + \gamma + \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\right)\right)$.

The theorem leads to the following corollary (extinction condition)

Corollary

Let assumptions (4) and (6) hold.

$$\text{If } (B \leq 0 \text{ and } C < 0) \text{ or } \left(B > 0 \text{ and } \frac{B^2}{4A} + C < 0 \right),$$

then for any initial value $I(0) \in (0, 1)$, the solution $I(t)$ of the stochastic differential equation (5) tends to zero exponentially almost surely. In other words, the disease dies out with probability one.

Computer simulation for extinction

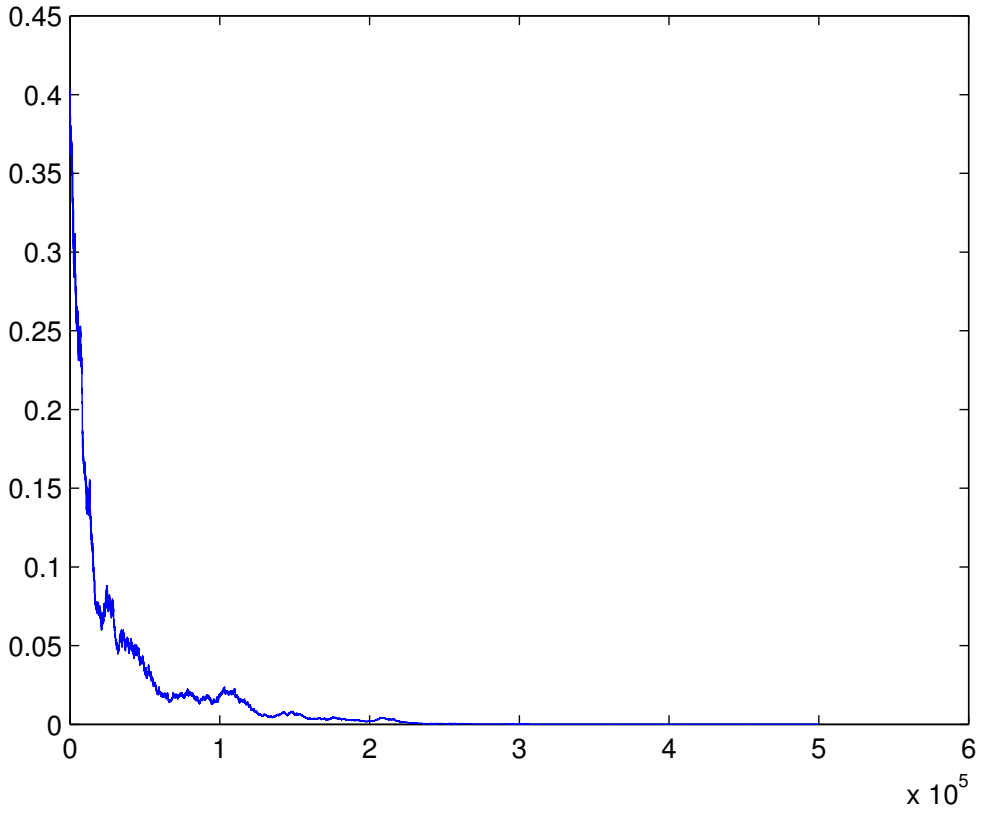

Figure: Computer simulation of a single path of $I(t)$ for the SDE model (5) with initial condition $I(0) = 0.4$ using the parameter values : $\mu = 0.014, \quad \beta = 0.31, \quad \gamma = 0.32, \quad \sigma = 0.11, \quad H(z) = 0.1,\ z \in \mathbb{R}$

Persistence of the disease

We consider the two following quadratic functions[1] :

$$Q(x) = -Ax^2 + Bx + C \text{and } Q'(x) = -A'x^2 + B'x + C'$$

where

$$A' = \frac{1}{2}\left(\sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz)\right) \geq A,$$
$$B' = -\beta + \sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz) \geq B,$$
$$C' = \beta - \left(\mu + \gamma + \frac{1}{2}\left(\sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz)\right)\right) \leq C.$$

[1] we already introduced A, B and C in 2

Theorem

Let assumptions (4) and (6) hold.
(i) If $0 \leq H(z) < 1$, $z \in \mathbb{Y}$ *and* $C' > 0$ *then,*

$$\limsup_{t \to \infty} I(t) \geq \xi', \quad a.s. \tag{7}$$

(ii) If $C > 0$ *then,*

$$\liminf_{t \to \infty} I(t) \leq \xi, \quad a.s., \tag{8}$$

where, ξ *and* ξ' *are the positives roots of* $Q(x) = 0$ *and* $Q'(x) = 0$ *respectively.*

Computer simulation for persistence

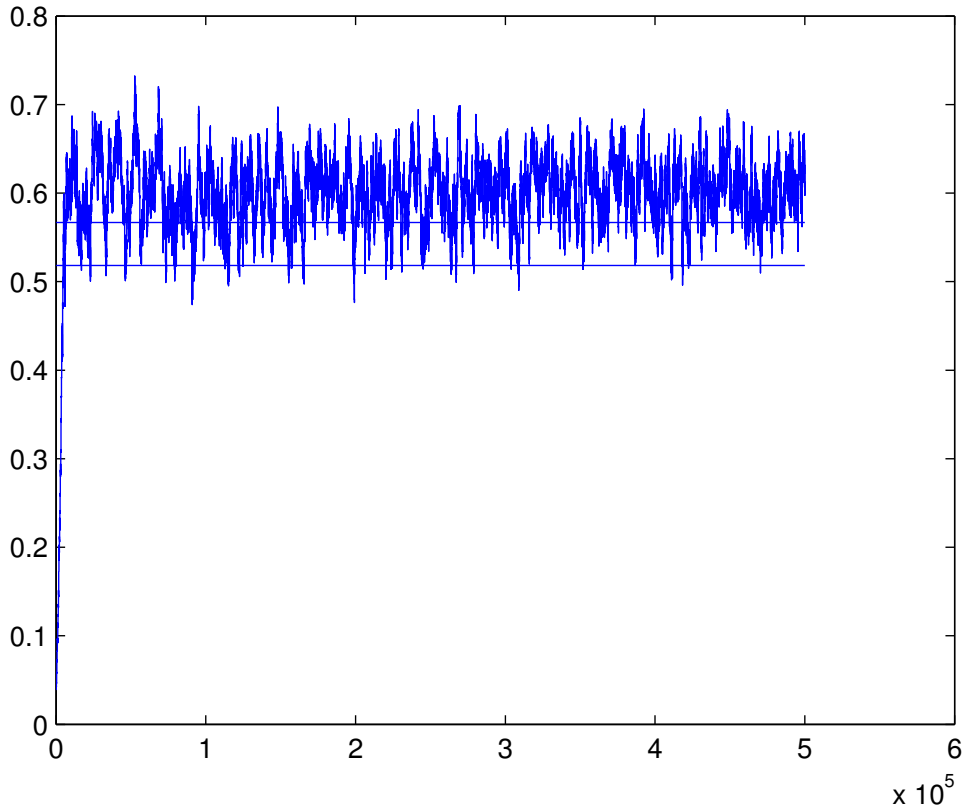

Figure: Results of one simulation run of SDE (5) with initial condition $I(0) = 0.4$ using the parameter values :
$\mu = 0.017, \quad \beta = 1.3, \quad \gamma = 0.5, \quad \sigma = 0.2, \quad H(z) = 0.95, \; z \in \mathbb{R}$

Conclusion

This work studied a stochastic SIS epidemic model with constant population size driven by Lévy noise. We first proved the existence and uniqueness of its global positive solution. Then, the long-term behavior of the stochastic SIS epidemic model is investigated. We obtained sufficient criteria for extinction and persistence of the epidemic in the population.


Thank you!
H.E.M

References I

1. V.Capasso, Mathematical structures of epidemic systems, Corrected 2nd printing, Springer, Heidelberg, 188 2008.
2. E. Beretta, Y. Takeuchi, 1995. Global stability of a SIR epidemic model with time delay. J. Math. Biol., 33, 250-260.
3. J. Zhou, H.W. Hethcote, Population size dependent incidence in models for diseases without immunity, J. Math. Biol. 32 (1994) 809-834.
4. A. Gray, D. Greenhalgh, X. Mao., J. Pan, The SIS epidemic model with Markovian switching, J. Math. Anal. Appl. 394 (2012) 496-516.
5. N. Dalal, D.Greenhalgh, X. Mao, A stochastic model of AIDS and condom use, J. Math. Anal. Appl. (2007) 325, 36-53.

References II

6. A. Lahrouz, L. Omari, D. Kiouach, A. Belmaati, Complete global stability for an SIRS epidemic model with generalized non-linear incidence and vaccination, Appl. Math. Comput. 218 (2012) 6519-6525.

7. H. W. Herbert, J. A. Yorke, Gonorrhea Transmission Dynamics and Control, Lecture Notes in Biomathematics, 56. Berlin: Springer-Verlag 1984.

8. A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A Stochastic Differential Equation SIS Epidemic Model, SIAM J. Appl. Math. 71 (2011) 876-902.

9. L. Wang, W. Yang. Global dynamics of a two-patch SIS model with infection during transport. Appl. Math. Comput., 217 (2011) 8458-8467.

References III

10. A. Lahrouz, A. Settati. Asymptotic properties of switching diffusion epidemic model with varying population size. Applied Mathematics and Computation, 219 (2013) 11134-11148.

11. X. Li, D. Jiang, X. Mao, Population dynamical behavior of Lotka-Volterra system under regime switching, J.Comput. App. Math. 232 (2009) 427-448.

12. J. Bao , X. Mao , G. Yin , C. Yuan; Competitive Lotka-Volterra population dynamics with jumps, Nonlinear Anal. 74 (2011) 6601-16.

13. J. Bao, C. Yuan, Stochastic population dynamics driven by LÃ©vy noise, J. Math. Anal. Appl. 391 (2012) 363-75.

14. Zhang, Xianghua, and Ke Wang. Stochastic SIR model with jumps. Appl. Math. Lett. 26 (2013) 867-874.

References IV

15. J. Bao, B. Böttcher, X. Mao, C. Yuan, Convergence rate of numerical solutions to SFDEs with jumps. J. Comput. Appl. Math. 236 (2011) 119-131.

16. R. Lipster, A strong law of large numbers for local martingales, Stochastics 3 (1980) 217-228.

17. C. Zhu, G. Yin, On competitive Lotka-Volterra model in random environments. J. Math. Anal. Appl. 357 (2009) 154-170.